Section 14.1

Functions of Two or More Variables

Prelecture Review Video

1 Domain and Range

Functions of Two Variables

A **function f of two variables** is a rule that assigns to each ordered pair (x, y) in a set D a unique number denoted by $f(x, y)$.

- The set D is called the **domain** of f.
- The **range** of f is the set of output values of f for points in D.

$$\text{Range}(f) = \{f(x, y) \mid (x, y) \in D\}$$

- The domain is a subset of $\mathbb{R}^2$, and the range is a subset of $\mathbb{R}$.

Often we write $z = f(x, y)$ for pairs (x, y) in D.

- x and y are the **independent** variables of f.
- z is the **dependent** variable of f.

Video

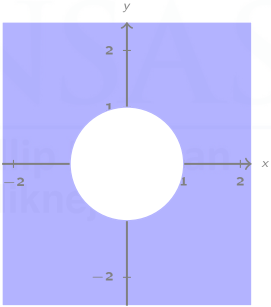
Functions of Two Variables

Example 1: Find the domain and range of the function

$$z = f(x, y) = \sqrt{x^2 + y^2 - 1}$$

Domain: The formula for z makes sense only if $x^2 + y^2 - 1 \geq 0$, so points in the domain satisfy the inequality $x^2 + y^2 \geq 1$. The domain consists of the unit circle and the region outside it.

Range: $[0, \infty)$

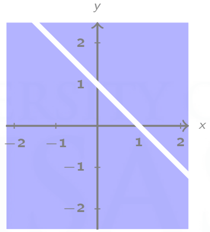


Example 2: Find the domain and range of the function

$$z = f(x, y) = \frac{1}{x + y - 1}.$$

Domain: $x + y - 1 \neq 0$, so points in the domain satisfy the equality $y \neq 1 - x$.

Range: $(-\infty, 0) \cup (0, \infty)$

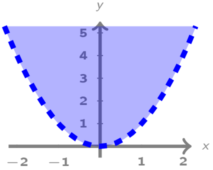


Example 3: Find the domain and range of the function

$$z = f(x, y) = \ln\left(y - x^2\right).$$

Domain: $y - x^2 > 0$, so points in the domain satisfy the inequality $y > x^2$.

Range: $(-\infty, \infty)$

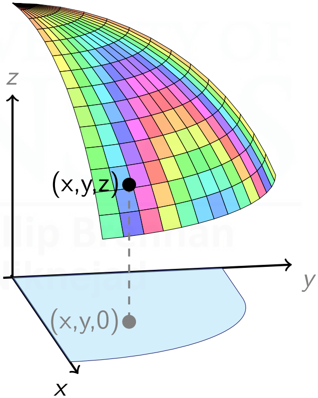


Graphing Functions of Two Variables

The **graph** of the function $z = f(x, y)$ with domain D is the set of points

$$\{(x, y, z) \mid z = f(x, y), (x, y) \in D\}$$

Note: The domain is the projection of the graph on xy-plane.



2 Graphs

Graphing Functions of Two variables

Graphs the functions in Examples 1,2, and 3.

$$f(x,y) = \sqrt{x^2 + y^2 - 1}$$

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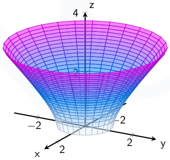
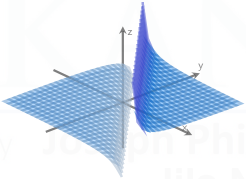
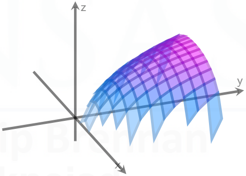
$$h(x,y) = \ln\left(y - x^2\right)$$







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$$g(x,y) = \frac{1}{x + y - 1}$$

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Functions of More Than Two Variables

We have introduced functions of two variables $z = f(x, y)$ where $(x, y) \in D \subset \mathbb{R}^2$.

- D is the domain of f (the set of possible input values);
- The range is $\{f(x, y) \,|\, (x, y) \in D\}$;
- The graph of f is $\{(x, y, z) \,|\, (x, y) \in D \text{ and } z = f(x, y)\}$.

In the same fashion, we can define functions of <u>three or more variables</u>.

$$w = f(x, y, z) \qquad \text{or} \qquad y = g(x_1, x_2, \ldots, x_n)$$

The domain and range of these functions are defined in the same way: the domain is the set of possible inputs and the range is the set of possible outputs. However, their graphs are difficult to visualize directly.

For functions of two or three variables, we can use **contour maps** as an alternative means of visualization.

3 Level Things

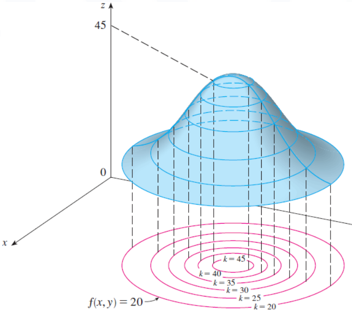
Level Curves and Contour Maps

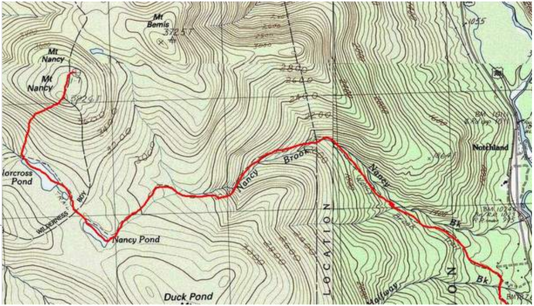
Given a function $z = f(x, y)$, the **level curve** with level k is

$$L_k(f) = \{(x, y) \in D \mid f(x, y) = k\}$$

$L_k(f)$ is generally a *curve* in the domain D.

A **contour map** is a collection of several level curves in D used to represent the 3-dimensional graph as a 2-dimensional figure.

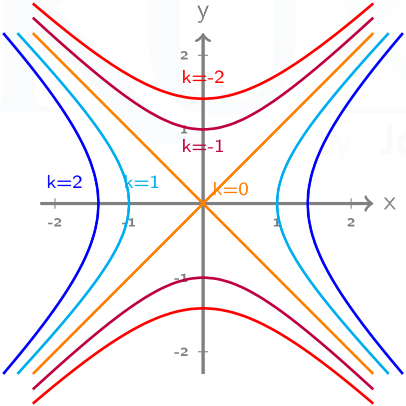


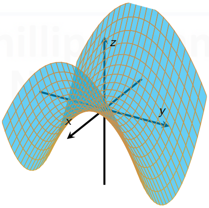


Example 4: Sketch a contour map of $z = f(x, y) = x^2 - y^2$ with levels $k \in \{0, \pm 1, \pm 2\}$.

<u>Solution:</u>

- $x^2 - y^2 = 0$ implies that $x = \pm y$.
- $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$ are hyperbolas opening horizontally.
- $x^2 - y^2 = -1$ and $x^2 - y^2 = -2$ are hyperbolas opening vertically.



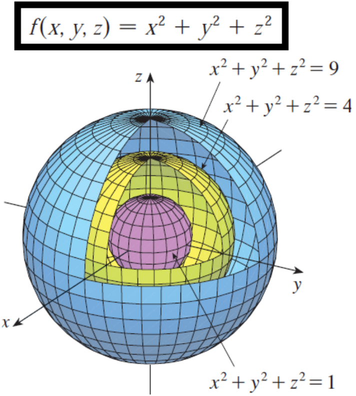


Level Surfaces

Let $w = f(x, y, z)$ be a function of three variables with domain D, and let k be a constant. The **level surface** with level k is

$$L_k(f) = \{(x, y, z) \mid f(x, y, z) = k\}$$

$L_k(f)$ is typically a surface in D, but can be something of lower dimensions.

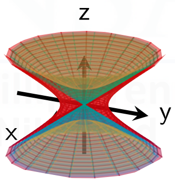


Level Surfaces

Example 5: Sketch a contour map for $w = f(x, y, z) = \frac{x^2}{4} + \frac{y^2}{2} - z^2$ with levels $k \in \{1, 0, -1\}$.

Solution:

k	$L_k(f)$	Description
1	$\frac{x^2}{4} + \frac{y^2}{2} - z^2 = 1$	Hyperboloid of one sheet
0	$\frac{x^2}{4} + \frac{y^2}{2} - z^2 = 0$	Cone
-1	$\frac{x^2}{4} + \frac{y^2}{2} - z^2 = -1$	Hyperboloid of two sheets



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